Ground-State Energy of a Two-Nucleon System*

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The ground-state energy of a system composed of an arbitrary number of nucleons coupled to a meson field is investigated. In the investigation, the nucleons are treated as an external source coupled to the meson field. Thus, only the mesons are treated as a quantum field and the ground-state energy is the energy of the zero-meson state. By using Fredholm determinants, the energy of the zero-meson state for the coupled source-field system is calculated relative to the energy of the zero-meson state for the noninteracting source-field system. The ground-state energy of this system is used to define the two-nucleon potential. This potential is calculated for two static nucleons coupled to a neutral scalar field and to a symmetric pseudoscalar field. In the case of the neutral scalar field, the well-known exact result is obtained. For the symmetric pseudoscalar field, the result is in good agreement with the potential studied by Gartenhaus.

I. INTRODUCTION

THE approach presented in this paper for solution of the problem of internucleon forces was inspired by a calculation of Wentzel[1] for the pair theory of nuclear forces. Schwinger[2] has done a similar calculation for a Dirac field in the presence of an external Maxwell field. The solution of the problem of two static nucleons given in this paper is similar to one offered by Frank,[3] but it differs in the formal apparatus used and in the treatment of the vacuum meson field.

We make use of Baker's analysis of the Fredholm determinant.[4] Baker has applied the determinantal methods proposed by Schwinger[2,5] to the pion-nucleon system and has used them to calculate the low-energy pion-nucleon scattering.

At this point we must state what we mean by a two-nucleon potential. This potential must be a function of the separation, spin, and isotopic spin variables of the nucleons that describes the interaction between the nucleons in such a way that we can use it for the potential energy term in the Schrödinger equation. The resulting Schrödinger equation must give the bound states and the low-energy scattering of the system. For this purpose, the potential must be independent of the meson-field variables and must not be a function of the energy of the nucleons. It should describe the interaction with sufficient accuracy to enable us to calculate the low-energy scattering of the nucleons, as well as the bound state properties of the two-nucleon system. Since our potential will be suitable only for describing the low-energy scattering data, the short-range part of the potential must be adjusted phenomenologically. This is usually done by inserting a strongly repulsive core.

Such a potential has been calculated by several authors, notably Brueckner and Watson[6] and Henley and Ruderman.[7] The method used by these authors included calculation of the field-theoretical scattering amplitude for two nucleons and substitution of this result into the Lippmann-Schwinger relation. The latter is an integral relation between the scattering amplitude and the effective potential. Thus, the potential is defined as the kernel that gives the proper scattering amplitude for two nucleons.

It should be mentioned that Brueckner and Watson use a more elaborate method of calculation, which includes many effects that occur in higher orders. Despite the difference in the Brueckner-Watson and Henley-Ruderman techniques, their results more or less agree. Both use the results of the Foldy-Dyson transformation, which gives the low-energy pion-nucleon interaction used in their calculations. Brueckner and Watson argue that the pair term is suppressed, so the interaction finally used is that between static nucleons with a gradient coupling to the pion field. The calculation includes only fourth-order terms; higher order terms are assumed to be of such short range that they lie well within the region of the phenomenological repulsive core.

There are a number of objections to the foregoing method of calculation. One objection is the uniqueness of the solution of the Lippmann-Schwinger equation. Also, it certainly is not to be assumed that the scattering data should ultimately yield all the bound-state properties of the two-nucleon system. In line with the uniqueness problem, there are ambiguities as to what types of Feynman graphs are to be included in the calculation.[7] Such a situation seems to call for a more precise definition of the potential and for an unambiguous procedure for calculating it.

As opposed to the foregoing method of calculating the two-nucleon potential, we define the energy of an external source coupled to the meson field. This energy is defined in such a manner that the source can be made

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[1] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).
[2] J. Schwinger, Phys. Rev. **94**, 1362 (1954).
[3] W. F. Frank, Ann. Phys. (N. Y.) **17**, 205 (1962).
[4] M. Baker, Ann. Phys. (N. Y.) **4**, 271 (1958).
[5] J. Schwinger, Phys. Rev. **93**, 615 (1954).
[6] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).
[7] E. M. Henley and M. A. Ruderman, Phys. Rev. **92**, 1036 (1953); D. Feldman, *ibid.* **98**, 1456 (1953).

up of any number of nucleons. It turns out that this definition allows us to derive the two-nucleon potential in a completely unambiguous way.

We find that the source-field energy is given by the two following additive terms: (1) the term due to shift in energy of the source because of the "dressing" effect of the mesons and (2) the term due to the shift in the vacuum energy of the meson field. The latter contribution arises from the shift in the energy levels of single-meson states when the meson field is coupled to the source; this term is the one studied by Wentzel[1] and Schwinger.[2] The term arising from dressing of the source is called ΔM, because it corresponds to the mass renormalization in the case where the source is a single nucleon. It is ΔM that gives the major contribution to the potential for the pion-nucleon system.

Although we do our calculation using a static source, care is taken to formulate the problem in such a way that recoil of the nucleons can be included.

II. ENERGY OF THE SOURCE-FIELD SYSTEM

The model that we concern ourselves with in the remainder of this paper is that of a meson field coupled to an external, classically prescribed source. Thus, there are no field equations of motion for the source. The possibility of a relativistic treatment of the nucleons is discarded from the start. However, we formulate the problem in such a way that the motion of the nucleons could be included, but without the possibility of the formation of nucleon pairs. In short, the nucleons may recoil, but the number of nucleons is strictly conserved. The Hamiltonian for such a system can be written in the form

$$H=H_0+H_1,$$

where

$$H_0=\tfrac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}[a^\dagger(k)a(k)+a(k)a^\dagger(k)]+M\bar{\psi}\psi \qquad (1)$$

and

$$H_1=\sum_{\mathbf{k}}[V(k)a(k)+V^\dagger(k)a^\dagger(k)]. \qquad (2)$$

This Hamiltonian is similar to the one used by Wick[8] for the scattering of pions by static nucleons. The $a(k)$ are the annihilation operators for the meson-field quanta, and they satisfy the commutation relations.

$$[a(k),a^\dagger(k')]=\delta_{\mathbf{kk}'},\quad [a(k),a(k')]=0. \qquad (3)$$

In the event that we wish to deal with pions specifically, k represents the set of variables $(\mathbf{k},\lambda)$, where $\mathbf{k}$ is the momentum vector and λ is the isotopic spin index of the pion annihilated by $a(k)$. $V(k)$ is the source current that may contain the variables of any number of nucleons. We have assumed a linear coupling of the mesons to the source, so $V(k)$ obeys the relations

$$V^\dagger(k)=[a(k),H_1],\quad [a(k),V(k')]=0. \qquad (4)$$

The second part of Eq. (4) is a property of linear coupling, but it is not an essential part of our formalism.

[8] G. C. Wick, Rev. Mod. Phys. **27**, 339 (1955).

In Eq. (1), ψ is a hypothetical field that represents the source and M is the energy of the noninteracting source. If the source happens to be n nucleons with the eigenvalues $p_1, p_2, \cdots, p_n$, then ψ can be represented by a product of the Heisenberg operators, $\psi_1(p_1)$, $\psi_2(p_2), \cdots, \psi_n(p_n)$, for the individual nucleons in the source. In this case, M is given by $E(p_1)+E(p_2)+\cdots+E(p_n)$, the sum of the individual nucleon energies.

The only quanta created or annihilated in the processes described by our Hamiltonian are the quanta of the meson field. The ground state of our system is considered to be the state with the source, but with no meson quanta present. For convenience, we refer to this state hereafter as the zero-meson state.

For H_0 of Eq. (1), representing the noninteracting source and field, the ground-state energy is given by

$$E_{0g}=\tfrac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}+M. \qquad (5)$$

E_{0g} is clearly infinite. The ground-state energy for the interacting system, which we call E_g, diverges in the same way. We might, however, expect the difference E_g-E_{0g} to be convergent if defined properly. It is this difference

$$E_v=E_g-E_{0g}$$

that we study.

In order to calculate the ground-state energy of the perturbed system, we assume that the Hamiltonian has been diagonalized and expressed in terms of a set of normal-coordinate field variables

$$H=\tfrac{1}{2}\sum_{\mathbf{k}}\Omega_{\mathbf{k}}[A^\dagger(k)A(k)+A(k)A^\dagger(k)]+(M+\Delta M)\bar{\Psi}\Psi. \qquad (6)$$

The $A(k)$ are the meson-field normal coordinates and satisfy the same commutation relations as the $a(k)$:

$$[A(k),A^\dagger(k')]=\delta_{\mathbf{kk}'},\quad [A(k),A(k')]=0. \qquad (7)$$

If we now assume the existence of a vacuum state $|\psi_0\rangle$ such that

$$A(k)|\psi_0\rangle=\Psi|\psi_0\rangle=0,$$

then the $A^\dagger(k)$ generate eigenstates of the interacting system and can be said to create "physical" mesons. Similarly, the $\bar{\Psi}$ operator can be thought of as creating the "clothed" source.

Assume that the coupling between the nucleons and the meson field is "turned on." The effect of the coupling is to shift all the energy levels away from their unperturbed values. ΔM is the energy shift of the source due to the dressing effect of the virtual meson cloud that appears when the coupling is turned on. Similarly, the $\Omega_{\mathbf{k}}$ are the shifted single-meson energies of the field.

Examples of the $A(k)$ operators are easily found. The $a_{\text{in}}(k)$ operators that arise in the problem of meson scattering are such operators. They annihilate physical mesons interacting with the source and create the eigenstates of the total Hamiltonian. It can also be shown that the Hamiltonian expressed in terms of the $a_{\text{in}}(k)$ operators is exactly of the form of Eq. (6). The normal field coordinates are by no means unique and may vary according to the boundary conditions on the problem. The $a_{\text{in}}(k)$ demonstrate that such normal coordinates exist, which is sufficient.

The ground-state energy of H is

$$E_g=\tfrac{1}{2}\sum_{\mathbf{k}}\Omega_{\mathbf{k}}+M+\Delta M \tag{8}$$

and

$$E_v=\tfrac{1}{2}\sum_{\mathbf{k}}(\Omega_{\mathbf{k}}-\omega_{\mathbf{k}})+\Delta M. \tag{9}$$

We have grouped the terms in Eq. (9) in such a way that we need only the sum over all the energy shifts of the mesons, plus the "renormalization term" for the source. We can now evaluate E_v by determining the "renormalization energy" for the source and the energy shifts for all the single-meson states. We use the symmetrized form for the Hamiltonian in Eqs. (1) and (6) in order to derive Eq. (9).

In the remainder of this discussion we shift the reference energy of the Hamiltonian by a constant and use the following Hamiltonian:

$$H_0' \quad \sum_{\mathbf{k}}\omega_{\mathbf{k}}a^\dagger(k)a(k), \tag{10}$$

$$H_1'=\sum_{\mathbf{k}}[V(k)a(k)+V^\dagger(k)a^\dagger(k)]-\Delta M. \tag{11}$$

Here the zero-meson state has zero energy, and the one-meson eigenvalues for H and H_0 are $\Omega_{\mathbf{k}}$ and $\omega_{\mathbf{k}}$, respectively. The variables for the source will be carried in the state vector.

Equation (10) is the normal product form of H_0, and it is the form most commonly used in literature dealing with low-energy meson physics. The normal product form of any of the dynamical variables has the property that the vacuum expectation value of the dynamical variables vanishes. The vacuum expectation value of the energy is just what we wish to calculate, so the normal product form, which subtracts off the vacuum energy, is not the appropriate energy operator for the derivation of Eq. (9). Once Eq. (9) is established, we can use Eqs. (10) and (11) to calculate $\Omega_{\mathbf{k}}$ and $\omega_{\mathbf{k}}$, which are now the energy eigenvalues of the single-meson states in the interacting and noninteracting systems, respectively.

A more general proof that the energy shift of the vacuum state of the meson field is $\frac{1}{2}\sum_{\mathbf{k}}(\Omega_{\mathbf{k}}-\omega_{\mathbf{k}})$ is given by Schwinger[2] in his treatment of the vacuum energy of fermions coupled to an external source.

III. THE FREDHOLM DETERMINANT

This section outlines the properties of the infinite determinants that are useful in calculating E_v. The material is based largely on the work of Baker,[3] and the reader is referred to his paper for a more complete discussion.

Formal Properties

$D(E)$ is defined by

$$D(E)=\det\left(\frac{E-H'}{E-H_0'}\right). \tag{12}$$

If the eigenvalues of H' and H_0' are represented by $E_{\mathbf{k}}$ and $E_{0\mathbf{k}}$, respectively, $D(E)$ can be written as

$$D(E)=\prod_k^u\left(\frac{E-E_k}{E-E_{0k}}\right). \tag{13}$$

If the system under consideration possesses certain symmetries, most of the eigenvalues present in Eq. (13) are degenerate. For example, the eigenstates of the various constants of the motion, such as linear and angular momentum are, in general, degenerate. In a field theory, there may be two states whose particle numbers differ but whose other eigenvalues, including their energy eigenvalues, are the same. For example, suppose we have a meson field with mass m. Above the threshold energy for two mesons, $E\geq 2m$, we may have states with either one or two mesons present and with the same eigenvalues for the other principal constants of the motion.

We denote by γ the eigenvalues of a set of observables that, together with the energy, are sufficient to form a complete set of observables for the system. An important observable present in the set is the physical, or "asymptotic," particle number. If we restrict ourselves to a collection of n asymptotically incoming or outgoing particles, we then say we are dealing with the subset γ_n only.

By a proper arrangement of the factors in Eq. (13), we can separate out the degeneracies from $D(E)$ and write

$$D(E)=\prod_\gamma\left[\prod_k\left(\frac{E-E_{\gamma k}}{E-E_{0\gamma k}}\right)\right]=\prod_\gamma D_\gamma(E). \tag{14}$$

In the $D_{\gamma_n}(E)$, each eigenvalue $E_{0\gamma k}$ appears only once, if at all.

The hermiticity of H' and H_0' implies that all the zeros and poles of $D_\gamma(E)$ lie along the real axis. For $D_{\gamma_n}(E)$ there is a continuous distribution of simple poles starting at nm along the positive real axis.

From Eq. (13) we conclude that $D_\gamma(E)$ is analytic in the E plane, which is cut along the real axis. Writing $D(E)$ in the form

$$D(E)=\det\left(1-\frac{1}{E-H_0'}H_1'\right) \tag{15}$$

indicates that each $D_\gamma(E)\to 1$ as $|E|\to\infty$.

Because there is no more than one factor of $(E-E_{0\gamma k})^{-1}$ in each of the $D_{\gamma_n}(E)$, we have the representation of

$$D_{\gamma_n}(E)=1+\sum_{E_{0k}=nm}^{\infty}\left(\frac{r_{\gamma_n k}}{E-E_{0k}}\right), \tag{16}$$

where $r_{\gamma_n k}$ are the residues of the simple poles of the $D_{\gamma_n}(E)$. It turns out that, for our purposes, the subdeterminant referring to the one-meson states is of primary interest. For the sake of simplifying a complicated notation, we suppress the index γ in the rest of this section. Accordingly, we may factor $D(E)$ into

$$D(E)=D_1(E)D_2(E)=\left(1+\sum_{E_{0k}=m}^{\infty}\frac{r_{1k}}{E-E_{0k}}\right)D_2(E). \quad (17)$$

$D_1(E)$ is the subdeterminant of $D(E)$ referring to one-meson states, and $D_2(E)$ involves at least two mesons.

Now, we wish to exhibit explicitly the dependence of $D(E)$ upon $(E-E_{0k})^{-1}$. To this end, we define

$$G_0^{\neq E_{0k}}(E)=\frac{1}{E-H_0'}(1-|E_{0k}\rangle\langle E_{0k}|), \quad (18)$$

where $|E_{0k}\rangle$ is an eigenstate of H_0' with eigenvalue E_{0k}.

In the following equation, we make use of the property that the determinant of the product of operators is the product of the determinants

$$\begin{aligned}D(E)&=\det[1-G_0^{\neq E_{0k}}(E)H_1'-(E-E_{0k})^{-1}|E_{0k}\rangle\langle E_{0k}|H_1']\\&=\det[1-G_0^{\neq E_{0k}}(E)H_1']\det\left\{1-\frac{|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}}{E-E_{0k}}\right\}.\end{aligned} \quad (19)$$

Performing the Fredholm expansion of the second factor in Eq. (19), we find

$$\begin{aligned}\det\left\{1-\frac{|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}}{E-E_{0k}}\right\}&=1-\frac{\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0k}\rangle}{E-E_{0k}}+\cdots\\&+\sum_{E_{01}\cdots E_{0n}}\frac{(-1)^n}{n!}\begin{vmatrix}\frac{\langle E_{01}|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{01}\rangle}{E-E_{0k}} & \cdots & \frac{\langle E_{01}|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-\prime}|E_{0n}\rangle}{E-E_{0k}}\\ \cdots & & \cdots\\ \frac{\langle E_{0n}|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{01}\rangle}{E-E_{0k}} & \cdots & \frac{\langle E_{0n}|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0n}\rangle}{E-E_{0k}}\end{vmatrix}.\end{aligned} \quad (20)$$

In Eq. (20), all terms except the first two vanish because all the minors have all their rows identical. We make use of the property that a determinant, with any pair of identical rows, vanishes. Thus,

$$\det\left\{1-\frac{|E_{0k}\rangle\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}}{E-E_{0k}}\right\}=1-\frac{\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|(E)_{0k}\rangle}{E-E_{0k}} \quad (21)$$

and

$$D(E)=\left\{1-\frac{\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0k}\rangle}{E-E_{0k}}\right\}D^{\neq E_{0k}}(E), \quad (22)$$

where

$$D^{\neq E_{0k}}(E)\equiv\det[1-G_0^{\neq E_{0k}}(E)H_1']. \quad (23)$$

In Eqs. (17) and (22), we have two equations showing explicitly the behavior of $D(E)$ about any pole $E_{0k}<2m$. We use these relations to calculate the residues r_{1k} of $D_1(E)$. To begin with, we expand the residue in Eq. (22) about E_{0k} and, comparing it with Eq. (17),

$$\begin{aligned}D(E)|_{E\to E_{0k}}=D^{\neq E_{0k}}(E_{0k})\Bigg\{1-&\frac{\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E_{0k})H_1']^{-1}|E_{0k}\rangle}{E-E_{0k}}\\&-(d/dE)\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0k}\rangle|_{E=E_{0k}}+O(E-E_{0k})\Bigg\}.\end{aligned} \quad (24)$$

Letting

$$Z_2=\left\{1-\frac{d}{dE}\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0k}\rangle|_{E=E_{0k}}\right\}^{-1},$$

neglecting everything of order $E-E_{0k}$ in Eq. (24), and combining the result with Eq. (17), we get

$$\left(1+\sum_{E_{0k'}}\frac{r_{1k'}}{E-E_{0k'}}\right)D_2(E)|_{E\to E_{0k}}=D^{\neq E_{0k}}(E_{0k})Z_2^{-1}+D^{\neq E_{0k}}\left\{\frac{\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E_{0k})H_1']^{-1}|E_{0k}\rangle}{E-E_{0k}}\right\}.$$

Equating like powers of $E-E_{0k}$ yields the two relations

$$D_2(E_{0k})r_{1k}=-D^{\neq E_{0k}}(E_{0k}) \times\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E_{0k})H_1']^{-1}|E_{0k}\rangle \quad (25)$$

and

$$D^{\neq E_{0k}}(E_{0k})Z_2^{-1}=D_2(E)D_1^{\neq E_{0k}}(E_{0k}). \quad (26)$$

Finally, we get

$$r_{1k}=-Z_2D_1^{\neq E_{0k}}(E_{0k}) \langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E_{0k})H_1']^{-1}|E_{0k}\rangle \quad (27)$$

as the expression for the residues of the subdeterminant $D_1(E)$.

In order to arrive at Eq. (27), we found it necessary to restrict E_{0k} to the range $E_{0k}<2m$, where $D_2(E)$ has no poles. Nevertheless, Eq. (27) is still valid in the region $E\geq 2m$ if we replace $|E_{0k}\rangle$ by $|E_{0k}\gamma_1\rangle$. This can be seen by the fact that the whole derivation goes through if we divide Eq. (22) and $D_2(E)$ by the factor

$$\prod_{\gamma\neq\gamma_1}\left\{1-\frac{\langle E_{0k}\gamma|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0k}\gamma\rangle}{E-E_{0k}}\right\}. \quad (28)$$

$|E_{0k}\gamma\rangle$ ranges over all the eigenstates of H_0' that are degenerate with the single-meson state in the region $2m\leq E<3m$. Thus, the modified equations have simple poles at $E=E_{0k}$. This procedure can be repeated for all regions $nm\leq E_{0k}<(n+1)m$. For more detail on this point see Ref. 4.

Construction of the Eigenstates of H'

If we write Eq. (13) in the form

$$D(E)=\prod_k\left(1-\frac{\Delta E_k}{E-E_{0k}}\right), \quad (29)$$

where ΔE_k is the shift in energy of the state $|E_{0k}\rangle$ when the coupling between the meson field and the source is turned on, and compare this with Eq. (22), the identification

$$\Delta E_k=E_k-E_{0k} =\langle E_{0k}|H_1'[1-G^{\neq E_{0k}}(E_k)H_1']^{-1}|E_{0k}\rangle \quad (30)$$

can be made by noticing that $D(E_k)=0$. Furthermore, if $|\psi_k\rangle$ is an eigenstate of the total Hamiltonian with eigenvalue E_k, and if it is normalized in such a way that $\langle E_{0k}|\psi_k\rangle=1$,

$$E_k-E_{0k}=\langle E_{0k}|H'-H_0'|\psi_k\rangle=\langle E_{0k}|H_1'|\psi_k\rangle. \quad (31)$$

This suggests

$$|\psi_k\rangle=[1-G_0^{\neq E_{0k}}(E_k)H_1']^{-1}|E_{0k}\rangle. \quad (32)$$

In fact,

$$\begin{aligned}H_0|E_{0k}\rangle &=H_0'[1-G_0^{\neq E_{0k}}(E_k)H_1'][1-G_0^{\neq E_{0k}}(E_k)H_1']^{-1}|E_{0k}\rangle,\\ &=H_0\left[1-\frac{1}{E_k-H_0'}(1-|E_{0k}\rangle\langle E_{0k}|)H_1'\right]|\psi_k\rangle,\\ &=H_0\frac{E_k-H_0'-H_1'}{E_k-H_0'}|\psi_k\rangle+E_{0k}\frac{\Delta E_k}{E_k-H_0'}|E_{0k}\rangle,\end{aligned}$$

and, consequently,

$$H_0(E_k-H_0')^{-1}(E_k-H_0'-H_1')|\psi_k\rangle =\left[1-\frac{\Delta E_k}{E_k-E_{0k}}\right]E_{0k}|E_{0k}\rangle=0, \quad (33)$$

so that

$$(E_k-H')|\psi_k\rangle=0, \quad (34)$$

and, thereby, Eq. (32) is verified.

With regard to Eqs. (32) and (34), it is interesting to note that $|\psi_k\rangle$ is that eigenstate of the total Hamiltonian that goes over into the eigenstate $|E_{0k}\rangle$ of H_0', as the interaction between the nucleons and meson field is removed. Usually, one can construct the eigenstates of H' with the eigenvalue E_{0k}, and relate it to the state $|E_{0k}\rangle$. However, such a state does not properly account for the shift in energy of the meson states as the coupling is turned on. From the derivation of Eq. (34), it is clear that the eigenvalue E_k is equal to $E_{0k}+\Delta E_k$, and that we obtain this energy shift by using the propagator $G_0^{\neq E_{0k}}(E)$, rather than $(E-H_0')^{-1}$.

We turn now to the problem of normalizing the states. The normalization is found from

$$\begin{aligned}N&=\langle\psi_k|\psi_k\rangle\\ &=\langle E_{0k}|[1-H_1'G_0^{\neq E_{0k}}(E_k)]^{-1} \times[1-G_0^{\neq E_{0k}}(E_k)H_1']^{-1}|E_{0k}\rangle\\ &=\langle E_{0k}|1+[1-H_1'G_0^{\neq E_{0k}}(E_k)]^{-1}H_1'G_0^{\neq E_{0k}}(E_k)^2H_1' \times[1-G_0^{\neq E_{0k}}(E_k)H_1']^{-1}|E_{0k}\rangle, \quad (35)\end{aligned}$$

or

$$N=Z_2^{-1} =1-(d/dE)\langle E_{0k}|H_1'[1-G_0^{\neq E_{0k}}(E)H_1']^{-1}|E_{0k}\rangle|_{E=E_k} \quad (36)$$

is the so-called state-vector-renormalization constant for the physical states. If one examines the Feynman diagrams that contribute to Z_2 in the expansion of Eq. (36), taking into account the special properties of $G_0^{\neq E_{0k}}(E)$, it is clear that Z_2 is the same no matter how many mesons are present in the state $|E_{0k}\rangle$. It is appropriate, therefore, to call Z_2 the renormalization of the source or the physical zero-meson state. Physically, Z_2 represents the probability of finding the bare source in the physical source state. Z_2 varies with the number of nucleons in the source. For example, If $Z_2^{(1)}$ is the state vector normalization constant for a source of one nucleon, it differs from the renormalization constant for two nucleons $Z_2^{(2)}$. They are related by the condition that

$$Z_2^{(2)} \to (Z_2^{(1)})^2,$$

as the separation of the two nucleons becomes infinite.

Finally, we examine the question of the energy shift due to the dressing of the source. In the case of one nucleon, this is just the mass renormalization. Assuming that we have adjusted the ground-state energy of H_0 so that it is zero, (i.e., let $H_0 \to H_0'$), we wish to do the same for the ground-state energy of H. From Eq. (30), we see that the energy shift of the zero-meson state is

$$\Delta E_0 = \langle 0|H_1|\psi_0\rangle, \tag{37}$$

where $|0\rangle$ and $|\psi_0\rangle$ are the noninteracting and physical states, respectively. By subtracting the c number, ΔE_0, from H_1, we achieve the renormalization of the source energy.

Hereafter, in analogy with the problem of mass renormalization, we refer to ΔE_0 as ΔM, and replace H_1 by $H_1 - \Delta M$:

$$\langle 0|H_1'|\psi_0\rangle = \langle 0|H_1 - \Delta M|\psi_0\rangle = 0. \tag{38}$$

Equation (38) now defines ΔM. Thus, by adopting the Hamiltonian of Eq. (11), we have removed, from $D(E)$, the pole at the origin.

Relationship of the Energy Shifts to the Observables

The results of this subsection are based primarily on physical arguments so that we may quickly develop some relationships important for the application of the properties of $D(E)$. These relationships are of fundamental importance and are derived in a more formal fashion in the literature.[4,9]

For the sake of simplicity, we can say that the aim of quantum mechanics is to calculate energy levels (or energy differences) and transition rates. Hence, we must show how the $D(E)$ is related to these quantities. We have already shown that the residues of the subdeterminants are related to the energy shifts, Eq. (27), so we will say no more about this matter. For the rest of the problem, it is perhaps simplest to establish the connection between the energy shifts and the phase shifts in a scattering problem, and thereby define the S matrix.

Let us consider the situation where we have perturbed and unperturbed waves, of type γ, in a spherical box. The relationship between the energy-level spacing dE and the energy shifts are shown in Fig. 1. The E_{0i} represent the unperturbed spectrum of the quasicontinuum, which ranges from m to infinity, while the E_i represent the corresponding perturbed spectrum. The manner in which the energy levels are shifted when the system is perturbed is illustrated by the ΔE_i. It is important to realize that the ΔE_i need not have the same value as one of the ΔE_{0i}. In other words, as long as we are dealing with waves in a box, the spectra of the perturbed and unperturbed waves are entirely different, as indicated in Fig. 1.

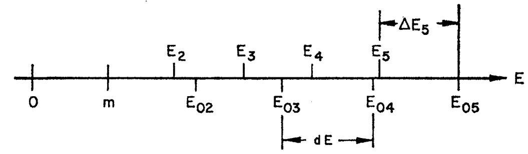

FIG. 1. Relationship between dE and ΔE_k.

The phases of the perturbed and unperturbed waves at large distances, r, from the center of the box are $k_\gamma^{(n)}R+\phi^\gamma$, and $k_{0\gamma}^{(n)}+\phi_0^\gamma$, respectively. We apply an arbitrary boundary condition on the phases at the surface of the box

$$k_\gamma^{(n)}R+\phi^\gamma = k_{0\gamma}^{(n)}R+\phi_0^\gamma = n\pi+\beta, \tag{39}$$

where the superscript n refers to a particular natural mode of the box. From Eq. (39), it follows that

$$dn = (R/\pi)dk \tag{40}$$

and that the phase shifts are given by

$$\delta_\gamma \equiv \phi^\gamma - \phi_0^\gamma = R(k_{0\gamma}^{(n)} - k_\gamma^{(n)}). \tag{41}$$

Now, if we write the energy shift in the form

$$\Delta E_{\gamma k} = \frac{dE}{dk_0}(k_\gamma^{(n)} - k_{0\gamma}^{(n)}) = -\frac{dE}{dk}\left(\frac{\delta_\gamma}{R}\right), \tag{42}$$

and make use of Eq. (40), we have

$$\Delta E_{\gamma k} = -(\delta_\gamma/\pi)dE. \tag{43}$$

Equation (43) not only gives the general connection between the energy shifts and the phase shifts but also shows the difference between the energy shifts and the energy-level spacing. Only when the energy-level spacing approaches zero as the radius of the box gets infinitely large, does the relation

$$\Delta E_{\gamma k} = dE = 0$$

hold. This relation is not true, of course, if we are dealing with a bound state.

IV. CALCULATION OF E_v

In this section we apply the properties of $D(E)$ to the calculation of E_v.

Vacuum Energy of the Meson Field

In order to calculate E_v, we need a way to sum all the energy shifts of the one-meson states. In order to accomplish this, let us consider a function $f(E)$, regular in the E plane. For such a function we can write the following relationship:

$$\sum_k [f(\Omega_k) - f(\omega_k)] = \frac{1}{2\pi i}\int_C dE\, f(E)\left(\frac{1}{E-\Omega_k} - \frac{1}{E-\omega_k}\right), \tag{44}$$

where C is the contour shown in Fig. 2.

[9] B. S. DeWitt, Phys. Rev. **103**, 1565 (1956).

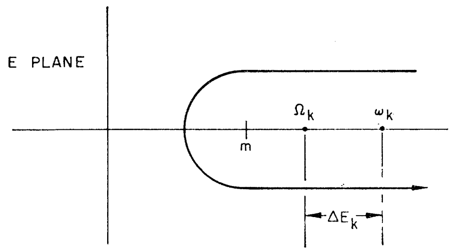

FIG. 2. The countour used in Eq. (44).

The function $D_1(E)$ can be expressed by

$$D_1(E)=\prod_k\left(\frac{E-\Omega_k}{E-\omega_k}\right), \tag{45}$$

where k is the label for a complete set of observables and ranges only over single-meson states. We can relate Eq. (44) to $D_1(E)$ in the following way:

$$\frac{dD_1(E)}{dE}=D_1(E)\sum_k\left(\frac{1}{E-\Omega_k}-\frac{1}{E-\omega_k}\right)$$

or

$$\frac{d}{dE}\ln D_1(E)=\sum_k\left(\frac{1}{E-\Omega_k}-\frac{1}{E-\omega_k}\right). \tag{46}$$

Equation (44) can now be written as

$$\sum_k\left[f(\Omega_k)-f(\omega_k)\right]=\frac{1}{2\pi i}\int_C dE\, f(E)\frac{d}{dE}\ln D_1(E). \tag{47}$$

Letting $f(E)=E$, and integrating Eq. (47) by parts, yields

$$\sum_k(\Omega_k-\omega_k)=-\frac{1}{2\pi i}\int_C dE\ln D_1(E)$$
$$=\frac{1}{2\pi i}\int_m^\infty dE\ln\frac{D_1(E+i\epsilon)}{D_1(E-i\epsilon)}. \tag{48}$$

We now factorize $D_1(E)$ into subdeterminants relating to different eigenvalues for single-meson states of a complete set of observables k, where k is the set (ω_k,γ_1), ω_k is the energy, and γ_1 represents the eigenvalues of the complete set excluding the energy:

$$D_1(E)=\prod_{\gamma_1}D_{\gamma_1}(E), \tag{49}$$

and

$$D_{\gamma_1}(E)=1+\sum_{\omega_k}\frac{r_{\gamma_1}(\omega_k)}{E-\omega_k}. \tag{50}$$

It is clear that $D_{\gamma_1}(E)$ has simple poles and that

$$D_{\gamma_1}(E^*)=D_{\gamma_1}(E)^*, \tag{51}$$

which allows us to write,

$$\mathrm{Im}D_{\gamma_1}(E+i\epsilon)=\frac{1}{2i}\left[D_{\gamma_1}(E+i\epsilon)-D_{\gamma_1}(E-i\epsilon)\right]$$
$$=-\pi\sum_{\omega_k}r_{\gamma_1}(\omega_k)\delta(E-\omega_k), \tag{52a}$$

and

$$\mathrm{Re}D_{\gamma_1}(E+i\epsilon)=\tfrac{1}{2}\left[D_{\gamma_1}(E+i\epsilon)+D_{\gamma_1}(E-i\epsilon)\right]$$
$$=1+P\sum_{\omega_k}\frac{r_{\gamma_1}(\omega_k)}{E-\omega_k}=D_{\gamma_1}{}^{\neq\omega_k}(E). \tag{52b}$$

Using Eq. (49) in Eq. (48), we find

$$\tfrac{1}{2}\sum_k(\Omega_k-\omega_k)=\frac{1}{4\pi i}\sum_{\gamma_1}\int_m^\infty dE\ln\frac{D_{\gamma_1}(E+i\epsilon)}{D_{\gamma_1}(E-i\epsilon)},$$
$$=\frac{1}{2\pi}\sum_{\gamma_1}\int_m^\infty dE\arctan\frac{\mathrm{Im}D_{\gamma_1}(E+i\epsilon)}{\mathrm{Re}D_{\gamma_1}(E+i\epsilon)}; \tag{53}$$

and from Eqs. (52a,b) and (27), we find

$$\frac{\mathrm{Im}D_{\gamma_1}(E+i\epsilon)}{\mathrm{Re}D_{\gamma_1}(E+i\epsilon)}=-\pi\frac{r_{\gamma_1}(E)}{D_{\gamma_1}{}^{\neq E}(E)}$$
$$=\pi Z_2\langle E\gamma_1|H_1'[1-G_0{}^{\neq E}(E)H_1']^{-1}|E\gamma_1\rangle. \tag{54}$$

Equation (54) finally allows us to write

$$\tfrac{1}{2}\sum_k(\Omega_k-\omega_k)=\frac{1}{2\pi}\sum_{\gamma_1}\int_m^\infty dE$$
$$\times\arctan\pi Z_2\langle E\gamma_1|H_1'[1-G_0{}^{\neq E}(E)H_1']^{-1}|E\gamma_1\rangle, \tag{55}$$

comparing Eq. (53) with Eq. (43), as follows:

$$\tfrac{1}{2}\sum_k(\Omega_k-\omega_k)=-\frac{1}{2\pi}\sum_{\gamma_1}\int_m^\infty dE\,\delta_{\gamma_1}(E), \tag{56}$$

so that we are led to the relation

$$(1/\pi)\tan\delta_{\gamma_1}(E)$$
$$=-Z_2\langle E\gamma_1|H_1'[1-G_0{}^{\neq E}(E)H_1']^{-1}|E\gamma_1\rangle. \tag{57}$$

This last result is in agreement with a well-known expression used for calculating phase shifts.[4]

The Fourth-Order Potential

If we collect the results of Eqs. (9) and (55), we get the following relation:

$$E_v=\Delta M+\frac{1}{2\pi}\sum_{\gamma_1}\int_m^\infty dE$$
$$\times\arctan\pi Z_2\langle E\gamma_1|H_1'[1-G_0{}^{\neq E}(E)H_1']^{-1}|E\gamma_1\rangle. \tag{58}$$

Considering the case of a meson field coupled to two static sources that have a separation x, we find that E_v is a function of the separation. The potential is

$$U(x)=E_v(x)-E_v(\infty),$$

and our problem now becomes one of evaluating $U(x)$ to fourth order in the coupling constant.

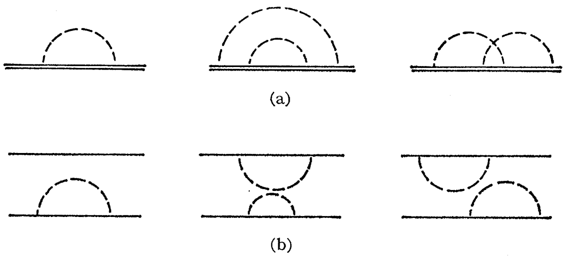

FIG. 3. Diagrams contributing to ΔM, up to fourth order.

Now, ΔM is obtained from Eq. (38) by expanding

$$\langle 0|(H_1-\Delta M)[1-G_0^{\neq 0}(0)(H_1-\Delta M)]^{-1}|0\rangle=0$$

to fourth order in H_1 and using the property that

$$G_0^{\neq\omega_k}(E)|\omega_k\rangle=0. \tag{59}$$

If, in addition, we have a source coupled linearly to the meson field, only those terms containing even powers of H_1 are retained. In this case, the fourth-order expansion of ΔM is

$$\Delta M=-\left(\sum_q\frac{V(q)V^\dagger(q)}{\omega_q}+\sum_{pq}\left[\frac{V(p)V(q)V^\dagger(p)V^\dagger(q)}{\omega_p(\omega_p+\omega_q)\omega_q}+\frac{V(q)V(p)V^\dagger(q)V^\dagger(q)}{\omega_q(\omega_p+\omega_q)\omega_q}\right]\right)\Big/\left(1+\sum_q\frac{V(q)V^\dagger(q)}{\omega_q^2}\right). \tag{60}$$

Equation (60) includes just the diagrams shown in Fig. 3(a). The horizontal double line represents the collection of nucleons constituting the source and should not be construed as necessarily representing a single nucleon. Perhaps a few words are in order regarding the fact that, in the two-nucleon case, diagrams corresponding to the fourth-order terms of Fig. 3(b) occur, whereas, for a single nucleon, no such modifications to the single-nucleon line occur in fourth order. The answer is found in the denominator, which is changed in such a way that it exactly compensates for the terms corresponding to the diagrams in question. Realization of this fact allows one to verify, by use of Eq. (60), that ΔM for two nucleons infinitely separated is equal to twice ΔM as calculated for a single nucleon.

It remains for us to evaluate the phase-shift terms up to fourth order using Eq. (57). The result is

$$\begin{aligned}\tan\delta_\gamma(\omega_k)=\pi\Bigg\{&2\frac{V(k)\Delta M^{(2)}V(k)}{\omega_k^2}-\sum_q\Bigg[\frac{V^\dagger(k)V(q)V^\dagger(q)V(k)}{\omega_k^2(\omega_k-\omega_q)}-\frac{V(q)V^\dagger(k)V^\dagger(q)V(k)}{\omega_k\omega_q(\omega_k-\omega_q)}\\&-\frac{V^\dagger(k)V(q)V(k)V^\dagger(q)}{\omega_k\omega_q(\omega_k-\omega_q)}+\frac{V(q)V^\dagger(k)V(k)V^\dagger(q)}{\omega_q^2(\omega_k+\omega_q)}-\frac{V(k)V(q)V^\dagger(q)V^\dagger(k)}{\omega_k^2(\omega_k+\omega_q)}-\frac{V(q)V(k)V^\dagger(q)V^\dagger(k)}{\omega_q\omega_k(\omega_k+\omega_q)}\\&-\frac{V(k)V(q)V^\dagger(k)V^\dagger(q)}{\omega_k\omega_q(\omega_k+\omega_q)}-\frac{V(q)V(k)V^\dagger(k)V^\dagger(q)}{\omega_q^2(\omega_k+\omega_q)}\Bigg]\Bigg\}\left[1+\sum_q\frac{V(q)V^\dagger(q)}{\omega_q^2}\right]^{-1},\end{aligned} \tag{61}$$

where $\Delta M^{(2)}$ is the second-order mass term.

In order to proceed further with the calculation of E_v, we must specify exactly the number of nucleons present in the source. In case we are dealing with two nucleons, $V(k)$ will be of the form

$$V(k)=V(k)^{(1)}+V(k)^{(2)}, \tag{62}$$

where $V^{(1)}$ and $V^{(2)}$ contain the variables of the first and second nucleons, respectively.

Neutral Scalar Field

The properties of the neutral scalar field coupled to a static source are well known.[10] The two-nucleon potential, in this case, can be found exactly, and we will use this result as a check against the present method. One important property of the neutral scalar field is that there is no scattering from a static source (the phase shifts are zero). Thus, according to Eq. (58), the only nonvanishing terms are those belonging to ΔM. Expanding the denominator of Eq. (60), retaining terms up to fourth order, and using the property that the $V(k)$ commute with each other,

$$E_v=-\textstyle\sum_q V(q)V^\dagger(q)/\omega_q. \tag{63}$$

In addition, the source current for a source consisting of two nucleons is

$$V(q)=(4\pi)^{1/2}N(f/m)[v(k)/(2\omega_{\mathrm{k}})^{1/2}]\times(e^{i\mathbf{k}\cdot\mathbf{x}_1}+e^{i\mathbf{k}\cdot\mathbf{x}_2}), \tag{64}$$

where $N=(\text{quantization volume})^{-1/2}$, f is the rationalized coupling constant, m the mass of the field quanta, $v(k)$ is the cutoff function for the nucleons, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are the positions of nucleons one and two, respectively. Letting $\mathbf{x}_1=-\mathbf{x}_2$, and substituting Eq. (64) into Eq. (63),

$$E_v=-\frac{1}{2\pi^2}\left(\frac{f}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^2}(1+\cos 2\mathbf{k}\cdot\mathbf{x}). \tag{65}$$

Equation (65) gives us the energy of the zero-meson state as a function of $2x$, the separation of the nucleons.

[10] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949).

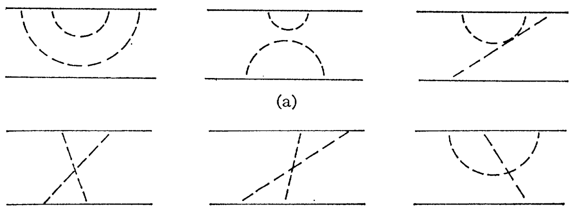

FIG. 4. Representative processes occurring in Eq. (71).

For infinite separation we have

$$E_v(\infty)=-\frac{1}{2\pi^2}\left(\frac{f}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^2}, \tag{66}$$

which must be interpreted as twice the renormalization energy of a nucleon, i.e., the sum of the self-energies of two noninteracting sources. The potential energy of two nucleons is defined so that it is zero when the nucleons have an infinite separation. We call $U(2x)$ the potential energy, so that

$$U(2x)=E_v(2x)-E_v(\infty)=\frac{1}{2\pi^2}\left(\frac{f}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^2}. \tag{67}$$

This is the well-known expression for the potential energy of two static nucleons coupled to the neutral scalar field.

V. CHARGED PSEUDOSCALAR FIELD

Proceeding along the lines that led to Eq. (67), we use

$$V(k\lambda)=(4\pi)^{1/2}iN\frac{f}{m}\frac{v(k)}{(2\omega_k)^{1/2}}\times(\boldsymbol{\tau}_\lambda^{(1)}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}}+\boldsymbol{\tau}_\lambda^{(2)}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}) \tag{68}$$

as the source current for two static nucleons coupled to the charged pseudoscalar field. The $\boldsymbol{\sigma}^{(i)}$ and $\boldsymbol{\tau}^{(i)}$ are the spin and isotopic spin indices, respectively, and the superscripts refer to the different nucleons.

The second-order term for ΔM is easily calculated:

$$\Delta M^{(2)}=-\frac{1}{2\pi^2}\left(\frac{f}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^2}\times(3q^2+\boldsymbol{\sigma}^{(1)}\cdot\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{q}\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\cos 2\mathbf{q}\cdot\mathbf{x}). \tag{69}$$

The x-dependent term is the second-order potential derived from ordinary perturbation theory. Carrying out the calculation for the fourth-order terms in ΔM, we find second- and fourth-order modifications of each nucleon line, one-meson exchange with modified vertex and nucleon line, and two-meson exchange. These processes are illustrated in Fig. 4. The evaluation of the fourth-order terms produces

$$\begin{aligned}\Delta M=-\Bigg\{&\frac{1}{2\pi^2}\left(\frac{f}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^2}(3q^2+\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{q}\cos 2\mathbf{q}\cdot\mathbf{x})\\ &+\frac{1}{4\pi^4}\left(\frac{f}{m}\right)^4\int\int d^3q\,d^3p\frac{|v(q)|^2|v(p)|^2}{\omega_q^3\omega_p^3}\Bigg[(3p^2q^2+p^2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{q}\cos 2\mathbf{q}\cdot\mathbf{x})(3\omega_p+\tfrac{1}{3}\omega_q)\\ &+\frac{p^2q^2}{\omega_p^2\omega_q^2}\left(\frac{4\omega_p^2}{\omega_p+\omega_q}-9\omega_q-4\omega_p\right)\Bigg]+\frac{1}{4\pi^4}\left(\frac{f}{m}\right)^4\int\int d^3q\,d^3p\frac{|v(q)|^2|v(p)|^2}{\omega_p^2\omega_q^3}\\ &\times\left[\left(\frac{3\omega_p}{\omega_p+\omega_q}+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\right)\boldsymbol{\sigma}^{(1)}\cdot\mathbf{p}\times\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{p}\times\mathbf{q}+(\mathbf{p}\cdot\mathbf{q})^2\left(3+\frac{2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\omega_p}{\omega_p+\omega_q}\right)\right]\cos 2(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}\Bigg\}\\ &\times\left\{1+\frac{1}{2\pi^2}\left(\frac{f}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^3}(3q^2+\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{q}\cos 2\mathbf{q}\cdot\mathbf{x})\right\}^{-1}.\end{aligned} \tag{70}$$

This expression for ΔM can be greatly simplified by following the renormalization procedure outlined by Chew[11] for our pseudovector coupling model. We notice that the denominator is just $Z_2^{(2)}$, as we have defined it for the two-nucleon source, evaluated to second order in f. The x-dependent part of $Z_2^{(2)}$ can be shown to be quite negligible as compared with the x-independent terms, at least in the region outside the repulsive core, say, $2x>0.5\ \mathrm{m}^{-1}$). In any case, the renormalized coupling constant is defined by

$$f_r=(Z_2^{(1)}/Z_1^{(1)})f. \tag{71}$$

Recalling that the x-independent part of $Z_2^{(2)}$ is just $(Z_2^{(1)})^2$, and noticing that the second term of ΔM supplies the vertex modifications required for $(Z_1^{(1)})^2$, we can replace f^2 appearing in Eq. (70) by f_r^2. To second order in the

[11] G. F. Chew, Phys. Rev. 94, 1748 (1954).

coupling constant, f_r^2 is given by

$$f_r^2=f^2\left[1-\frac{4}{3\pi^2}\left(\frac{f}{m}\right)^2\int d^3k\frac{k^2|v(k)|^2}{\omega_k^3}\right]. \tag{72}$$

From the way we defined the zero-meson state, it is understandable that our renormalized coupling constant should depend upon the separation of the two nucleons. It is also reassuring that f_r, in the limit of large x, should be very nearly equal to the renormalized coupling constant obtained for meson-nucleon scattering. The remainder of the second terms in Eq. (70) represents the propagator modification for the second-order term. The effect of such modifications has been estimated by Brueckner and Watson[6] to be small and, accordingly, we neglect it.

The contribution of ΔM to the two-nucleon potential is found to be

$$\begin{aligned}U(2x)=&-\frac{1}{2\pi^2}\left(\frac{f_r}{m}\right)^2\int d^3q\frac{|v(q)|^2}{\omega_q^2}\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\cdot\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{q}\cos 2\mathbf{q}\cdot\mathbf{x}\\&-\frac{1}{4\pi^4}\left(\frac{f_r}{m}\right)^4\int\int d^3pd^3q\frac{|v(q)|^2|v(p)|^2}{\omega_q^3\omega_p^2}\Bigg[\left(\frac{3\omega_p}{\omega_p+\omega_q}+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\right)\boldsymbol{\sigma}^{(1)}\cdot\mathbf{p}\times\mathbf{q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{p}\times\mathbf{q}\\&+(\mathbf{p}\cdot\mathbf{q})^2\left(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\frac{\omega_p}{\omega_p+\omega_q}\right)\Bigg]\cos 2(\mathbf{p}+\mathbf{q})\cdot\mathbf{x},\end{aligned} \tag{73}$$

which is just the potential studied by Gartenhaus.[12]

To Eq. (73), we must add the effect of the shift of the energy of the pion field using Eqs. (55) and (61). Calculation of this term is exceedingly difficult, but we can estimate the general effect without too much labor. If we denote the contribution to the potential of these terms, by $\Delta U(2x)$, we have

$$\Delta U(2x)=\frac{1}{2\pi}\int d^3k\times\arctan\pi\left[\frac{Z_2(x)\sum_q(x)-Z_2(\infty)\sum_q(\infty)}{1+\pi^2Z_2(x)Z_2(\infty)\sum_q(x)\sum_q(\infty)}\right], \tag{74}$$

where $\sum_q(x)$ represents the x-dependent part of Eq. (61). A simple calculation shows us that $\sum_q(\infty)$ is always several orders of magnitude greater than the x-dependent terms in the region outside the repulsive core ($2x>0.5\ \mathrm{m}^{-1}$). This occurs primarily because the sum over q brings in terms proportional to e^{-2mx} and e^{-2ax}, where a is the cutoff momentum. Thus, the contribution of Eq. (74) is quite negligible compared with Eq. (73).

VI. CONCLUSION

We have used the formalism of Fredholm determinants to calculate the two-nucleon potential. The resulting expression is very similar to the Gartenhaus potential. The main difference is that the potential is developed as the ratio of two power series in the coupling constant, unlike the perturbation expansion. This results in the renormalized coupling constant having a small dependence of x.

One formal advantage shows itself in the use of the propagator $G_0^{\neq E_0}(E)$, which leads to the exclusion of the troublesome "ladder" diagrams. The inclusion of such terms seems to be necessary when using a formalism based on the Lippmann-Schwinger equation, as well as in a Tamm-Dancoff treatment of the two-nucleon system, as was pointed out by Feldman.[7] Indeed, Henley and Ruderman found that they were necessary in the neutral scalar theory for removal of the fourth-order terms. We have found that the "ladder" terms may be excluded, and that the correct expression for the potential in neutral scalar theory also may be obtained by using the formalism of determinants.

The fact that the procedure developed in this paper is a well-defined procedure for calculating the energy of a source composed by arbitrary numbers of nucleons should make it suitable for study of the triton or He^3. Wentzel applied the method of determinants to the study of the energy of a proton lattice interacting through the pair theory.[1] A similar study for a source composed of many nucleons could yield information regarding the size of the two-body interaction relative to the many-body interactions.

ACKNOWLEDGMENT

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[12] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).